MATB44 Week 8 Notes

1. Review of linear algebra:

a) System of eqns:

- Consider the following:

$$\begin{aligned} a_{11}x_1 &+ a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 &+ a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ &\vdots \\ a_{n1}x_1 &+ a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{aligned}$$

We can write it in matrix form:

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\bar{x}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}_{\bar{b}}$$

in variable form

Note: All vectors will have a small horizontal line above.
I.e. $\bar{x}$ means that x is a vector.

We wrote the original system of eqns into this form: $A\bar{x} = \bar{b}$ where
- A is a matrix of the coefficients
- $\bar{x}$ is a vector of the variables/unknowns.
- $\bar{b}$ is a vector of the answers.

E.g. 1 Convert the following system of eqns to $A\bar{x} = \bar{b}$ form.

$$3x_1 + 2x_2 = 1$$
$$4x_1 - 7x_2 = 2$$

Soln:

$A = \begin{bmatrix} 3 & 2 \\ 4 & -7 \end{bmatrix}$ $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} 3 & 2 \\ 4 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

- Given a system of equations, we will have 1 of 3 possibilities for the number of solns:

a) No soln

b) Exactly 1 Soln

c) Infinitely many solns

- A system of eqns is called homogeneous if all the values in $\bar{b}$ are 0 and non-homogeneous if there is at least 1 non-zero value in $\bar{b}$.

- If we have a homogeneous system of eqns, we are guaranteed to have the trivial soln which is $x_1 = x_2 = \ldots = x_n = 0$.

I.e. The trivial soln occurs when all the values in $\bar{x} = 0$.

- For a homogeneous system of eqn, we will have 1 of 2 possibilities for the number of solns:
a) Exactly one soln, the trivial soln
b) Infinitely many non-trivial solns in addition to the trivial soln.

b) The determinant of a matrix:
- Recall that the determinant only works for square matrices, so assume that all matrices in this section are square matrices, unless otherwise stated.
- Denoted as det(A) where A is a square matrix.
- If $\det(A) = 0$, then A is singular while if $\det(A) \neq 0$, then A is nonsingular.
- Given a system of eqns:
a) If A is nonsingular, there will be exactly 1 soln to the system.
b) If A is singular, there will either be no soln or infinitely many solns to the system.
- Given a homogeneous system of eqns:
a) If A is non-singular, then there will be exactly 1 soln, the trivial soln.
b) If A is singular, there will be infinitely many solns.
- Given a matrix A:
a) If A is non-singular, then the vectors in A are linearly independent.
b) If A is singular, then the vectors in A are linearly dependent.

c) Eigenvalues and eigenvectors:

- Denoted as $A\bar{x} = \lambda\bar{x}$ where

a) A is a matrix

b) $\bar{x}$ is the eigenvector.

c) λ is the eigenvalue.

- If we have $A\bar{x} = \lambda\bar{x}$, then

  $\rightarrow A\bar{x} = \lambda I\bar{x}$ (where I is the identity matrix)

  $\rightarrow A\bar{x} - \lambda I\bar{x} = 0$

  If $\bar{x}$ is non-zero, we can solve for λ using the determinant.

  $\rightarrow |A - \lambda I| = 0$

Note: $\det(A - \lambda I) = 0$ is called the characteristic eqn.

E.g. 2 Find the eigenvalues and eigenvectors of the given matrix

$$A = \begin{bmatrix} 2 & 7 \\ -1 & -6 \end{bmatrix}$$

Soln:

$\det(A - \lambda I) = 0$

$$\begin{vmatrix} 2-\lambda & 7 \\ -1 & -6-\lambda \end{vmatrix} = 0$$

$(2-\lambda)(-6-\lambda) - (-1)(7) = 0$

$-12 - 2\lambda + 6\lambda + \lambda^2 + 7 = 0$

$\lambda^2 + 4\lambda - 5 = 0$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-4 \pm 6}{2} \rightarrow \lambda_1 = -5, \ \lambda_2 = 1$$

Now we will find the eigenvector for each eigenvalue.

$\lambda_1 = -5$

$$\left[\begin{array}{cc|c} 2-(-5) & 7 & 0 \\ -1 & -6-(-5) & 0 \end{array}\right] \begin{array}{l} R_1 \text{ (Row 1)} \\ R_2 \text{ (Row 2)} \end{array}$$

$$\left[\begin{array}{cc|c} 7 & 7 & 0 \\ -1 & -1 & 0 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \end{array}$$

Divide R_1 by 7 and R_2 by -1.

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \end{array}$$

$R_2 - R_1$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 + x_2 = 0 \rightarrow x_1 = -x_2$
$\rightarrow x_2 = -x_1$

$$\overline{x^1} = \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$

Notice that there is an infinite number of solns.
Furthermore, notice that the rows of $A - \lambda I$ are linearly dependent.
These are expected as $A - \lambda I$ is singular.

$\lambda_2 = 1$

$$\left[\begin{array}{cc|c} 2-1 & 7 & 0 \\ -1 & -6-1 & 0 \end{array}\right] \begin{array}{l} R_1 \text{ (Row 1)} \\ R_2 \text{ (Row 2)} \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 7 & 0 \\ -1 & -7 & 0 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \end{array}$$

Do $R_2 + R_1$

$$\left[\begin{array}{cc|c} 1 & 7 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 + 7x_2 = 0 \Rightarrow x_1 = -7x_2$

$$\overline{x^2} = \begin{bmatrix} -7x_2 \\ x_2 \end{bmatrix}$$

$$\overline{x^1} = \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}, \quad \overline{x^2} = \begin{bmatrix} -7x_2 \\ x_2 \end{bmatrix}$$

$\overline{x^1}$ and $\overline{x^2}$ are linearly dependent.

2. Systems of Linear Eqns With Constant Coefficients:
- Has the form $\overline{x'} = A\bar{x} + \bar{g}$

Note: We say the system is homogeneous if $\bar{g} = \bar{0}$ and non-homogeneous if $\bar{g} \neq \bar{0}$.
- We will focus on homogeneous systems first.

3. Homogeneous Systems:
- Has the form $\overline{x'} = A\bar{x}$

- Let $\bar{x} = e^{rt}\bar{z}$, where $\bar{z}$ is a vector.

$\bar{x}' = re^{rt}\bar{z}$

$A\bar{x} = e^{rt}A\bar{z}$

$x' = A\bar{x}$

$r\cancel{e^{rt}}\bar{z} = \cancel{e^{rt}}A\bar{z}$

$r\bar{z} = A\bar{z}$ ← Eigenvector eqn

$(A - rI)\bar{z} = 0$

Note: The system has a non-trivial soln iff $\det(A - rI) = 0$.

- A homogeneous eqn has unique, non-trivial solns iff the $\det(A - rI)$ is 0.

- Since the characteristic eqn will use the quadratic formula, we have 3 cases:

1. 2 real, distinct eigenvalues
2. Repeated eigenvalues
3. Complex eigenvalues

Note: In all cases, we will have 2 eigenvalues and 2 eigenvectors.

Note: Each eigenvalue will have an eigenvector.

Case 1: 2 real, distinct eigenvalues

E.g. 3 Solve $\bar{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\bar{x}$

Soln:

$$\begin{vmatrix} -2-r & 1 \\ 1 & -2-r \end{vmatrix}$$

$= (-2-r)^2 - 1$

$= 4 + 4r + r^2 - 1$

$= r^2 + 4r + 3 \rightarrow r_1 = -3, r_2 = -1$

$(A - rI)\bar{z} = 0$ ← Called Eigenvector Eqn

$$\begin{bmatrix} -2-(-3) & 1 \\ 1 & -2-(-3) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \leftarrow \text{When } r_1 = -3$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$z_1 + z_2 = 0$

$z_1 + z_2 = 0$ ← Recall that if matrix A is non-singular, the rows of A are linearly dependent. Hence, we will always get a redundant term.

$z_1 = -z_2$

Let $z_1 = 1. \Rightarrow z_2 = -1$

Eigenvector $= \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ ← Call this $\overline{z^1}$.

When $r_2 = -1$

$$\begin{bmatrix} -2-(-1) & 1 \\ 1 & -(2)-(-1) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-z_1 + z_2 = 0$

$z_1 - z_2 = 0$ ← Redundant

$z_1 = z_2$

Let $z_1 = 1. \Rightarrow z_2 = 1.$

Eigenvector $= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ← Call this $\overline{z^2}$.

Note: If you have 2 diff eigenvalues, you have 2 diff, linearly independent eigenvalues.

$$\overline{x} = C_1 e^{r_1 t}\,\overline{z^1} + C_2 e^{r_2 t}\,\overline{z^2}$$

$$= C_1 e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

E.g. 4 Solve $\overline{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\overline{x}$

Soln:

$$\begin{vmatrix} 1-r & 1 \\ 4 & 1-r \end{vmatrix}$$

$= (1-r)^2 - (1)(4)$

$= r^2 - 2r + 1 - 4$

$= r^2 - 2r - 3$

$r_1 = 3,\ r_2 = -1$

When $r = 3$

$$\begin{bmatrix} 1-r & 1 \\ 4 & 1-r \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-2z_1 + z_2 = 0$

$4z_1 - 2z_2 = 0$ ← Redundant

$2z_1 = z_2$

Let $z_1 = 1. \rightarrow z_2 = 2$

$\overline{z^1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ← Eigenvector

When $r = -1$

$$\begin{bmatrix} 1-r & 1 \\ 4 & 1-r \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2z_1 + z_2 = 0$
$4z_1 + 2z_2 = 0$ ← Redundant
$2z_1 = -z_2$
Let $z_1 = 1. \rightarrow z_2 = -2.$

$$\overline{z^2} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\begin{aligned} \overline{x} &= C_1 e^{r_1 t} \overline{z^1} + C_2 e^{r_2 t} \overline{z^2} \\ &= C_1 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \end{aligned}$$

E.g. 5 Solve $\overline{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \overline{x}$

Soln:

$$\begin{aligned} \begin{vmatrix} 3-r & -2 \\ 2 & -2-r \end{vmatrix} &= (3-r)(-2-r) - (-4) \\ &= -6 - 3r + 2r + r^2 + 4 \\ &= r^2 - r - 2 \end{aligned}$$

$$\begin{aligned} r &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{1 \pm 3}{2} \\ &= 2 \text{ or } -1 \end{aligned}$$

When $r=2$

$$\begin{bmatrix} 3-r & -2 \\ 2 & -2-r \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$z_1 - 2z_2 = 0$

$2 - 4z_2 = 0$ ← Redundant

$z_1 = 2z_2$

Let $z_1 = 1.$ $z_2 = 2.$

$$\overline{z^1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

When $r=-1$

$$\begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$4z_1 - 2z_2 = 0$

$2z_1 - z_2 = 0$ ← Redundant

$2z_1 - z_2 = 0$

$2z_1 = z_2$

Let $z_1 = 2.$ $z_2 = 1$

$$\overline{z^2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\overline{x} = c_1 e^{r_1 t} \overline{z^1} + c_2 e^{r_2 t} \overline{z^2}$$

$$= c_1 e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Case 2: Repeated Eigenvalues

E.g. 6 Solve $\overline{x}' = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \overline{x}$

Soln:

$$\begin{vmatrix} 1-r & -4 \\ 4 & -7-r \end{vmatrix} = (1-r)(-7-r)+16$$
$$= -7-r+7r+r^2+16$$
$$= r^2+6r+9$$
$$= (r+3)^2$$

$r_1 = r_2 = -3$

When $r = -3$

$$\begin{bmatrix} 1+3 & -4 \\ 4 & -7+3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$4z_1 - 4z_2 = 0$

$z_1 = z_2$

Let $z_1 = 1$. $z_2 = 1$.

$$\overline{z^1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

To find $\overline{z^2}$, we need a generalized eigenvector.

$x_1 = e^{-3t}\,\overline{z^1}$

$x_2 = te^{-3t}\,\overline{z^1} + e^{-3t}\,\overline{p}$, where p is an unknown vector.

Recall: $(A - rI)\bar{z} = 0$ is called the eigenvector eqn.

$(A - rI)\bar{p} = \bar{z}$ is called the generalized eigenvector eqn.

$$\begin{bmatrix} 1+3 & -4 \\ 4 & -7+3 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$4P_1 - 4P_2 = 1$
$4P_1 - 4P_2 = 1$
$P_1 - P_2 = 1/4$
$P_1 = 1/4 + P_2$

We can choose a few diff values for P_1

1. $P_1 = 1 \rightarrow P_2 = 3/4$
2. $P_2 = 0 \rightarrow P_2 = -1/4$

Note: We can only let $x_i = 0$ if we have a non-homogeneous eqn. If we have a homogeneous eqn, we can't.

See what happens when (1) - (2).

$$\begin{bmatrix} 1 \\ \ \end{bmatrix} - \begin{bmatrix} 0 \\ -1/4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

3/4 $\qquad = \bar{z}^1$

Note: If we get 2 different $\bar{p}$'s, their difference will be proportional to $\bar{z}$.

Let's use $\bar{p} = \begin{bmatrix} 0 \\ -1/4 \end{bmatrix}$

$\overline{x} = te^{-3t}\overline{z'} + e^{-3t}\overline{p}$ is a new soln.

Proof:

$\overline{x'} = e^{-3t}\overline{z'} - 3te^{-3t}\overline{z'} - 3e^{-3t}\overline{p}$

$A\overline{x} = A(te^{-3t}\overline{z'} + e^{-3t}\overline{p})$

$= te^{-3t}A\overline{z'} + e^{-3t}A\overline{p}$

Recall that $(A - rI)\overline{z'} = 0$ and that $r = -3$.

Hence, $(A - (-3))\overline{z'} = 0$

$\rightarrow A\overline{z} = -3\overline{z}$

Similarly, recall that $(A - rI)\overline{p} = \overline{z'}$ and that $r = -3$.

Hence, $(A - (-3))\overline{p} = \overline{z'}$

$\rightarrow A\overline{p} = -3\overline{p} + \overline{z'}$

Now, we get

$A\overline{x} = te^{-3t}(-3\overline{z'}) + e^{-3t}(-3\overline{p} + \overline{z'})$

$= -3te^{-3t}\overline{z'} - 3\overline{p}e^{-3t} + \overline{z'}e^{-3t}$

$= x'$

Note: You do not need to show this proof on tests/exam/quizzes/assignments etc.

Note: $\overline{z'}$ and $\overline{p}$ are linearly independent. We can prove this by showing that their determinant $\neq 0$. However, we don't need to prove it and can just state it.

Since $\overline{z'}$ and $\overline{p}$ are linearly indep, we have 2 linearly indep solns.

$\overline{x} = C_1e^{-3t}\overline{z'} + C_2(te^{-3t}\overline{z'} + e^{-3t}\overline{p})$

Case 3: Complex Eigenvalues

Eg. 7 Solve $\bar{x}' = \begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix}\bar{x}$

Soln:

$$\begin{vmatrix} 1-r & 2 \\ -5 & -1-r \end{vmatrix} = (1-r)(-1-r)+10$$
$$= -1-r+r+r^2+10$$
$$= r^2+9$$

$r^2+9=0$
$r^2=-9$
$r = \pm 3i$ ← Complex eigenvalues

Note: When we have complex eigenvalues, we also have complex eigenvectors.

$(A-rI)\bar{z}=0$

When $r=3i$

$$\begin{bmatrix} 1-3i & 2 \\ -5 & -1-3i \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(1-3i)z_1 + 2z_2 = 0$
$-5z_1 + (-1-3i)z_2 = 0$ ← Still redundant

Let $z_1 = 1$. $z_2 = -\frac{1}{2} + \frac{3}{2}i$

$$\bar{z} = \begin{bmatrix} 1 \\ -1/2 + 3i/2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ -1/2 \end{bmatrix} + i\begin{bmatrix} 0 \\ 3/2 \end{bmatrix}$$

$$\bar{x} = e^{3it}\,\bar{z}$$

$$= (\cos(3t) + i\sin(3t))\left(\begin{bmatrix} 1 \\ -1/2 \end{bmatrix} + i\begin{bmatrix} 0 \\ 3/2 \end{bmatrix}\right)$$

$$= \cos(3t)\begin{bmatrix} 1 \\ -1/2 \end{bmatrix} - \sin(3t)\begin{bmatrix} 0 \\ 3/2 \end{bmatrix} \leftarrow \text{Real part}$$

$$+ i\left(\cos(3t)\begin{bmatrix} 0 \\ 3/2 \end{bmatrix} + \sin(3t)\begin{bmatrix} 1 \\ -1/2 \end{bmatrix}\right) \leftarrow \text{Imaginary part}$$

General Soln:

$$c_1\left(\cos(3t)\begin{bmatrix} 1 \\ -1/2 \end{bmatrix} - \sin(3t)\begin{bmatrix} 0 \\ 3/2 \end{bmatrix}\right) +$$

$$c_2\left(\cos(3t)\begin{bmatrix} 0 \\ 3/2 \end{bmatrix} + \sin(3t)\begin{bmatrix} 1 \\ -1/2 \end{bmatrix}\right)$$